298. Proposed by C. N. SCHMALL, New York City.

A person desires to throw a stone so as to strike the greatest possible blow at a point in a smooth vertical wall at a height h above the ground. If his strength is sufficient to throw the stone vertically upwards to a height $2h$, show that he must stand at a distance $2h$ from the wall.

(The resistance of the air and the height of the hand are not taken into account.)

SOLUTION BY PAUL CAPRON, United States Naval Academy.

As the wall is smooth, the force of the blow is proportional to the square of the horizontal velocity of the stone, which in turn is the product of the given initial speed of the stone, v_0, and $\cos\alpha$, if α is the angle of elevation at the start. The problem is therefore to make $\cos\alpha$ a maximum, or to make α a minimum. $(0 < \alpha < \pi/2.)$ Let the required distance from the wall be z; then

$$z = v_0 \cos\alpha t, \qquad h = v_0 \sin\alpha t - \tfrac{1}{2}gt^2;$$

whence

$$2hv_0^2 = 2v_0^2 z \tan\alpha - gz^2 \sec^2\alpha.$$

Differentiating,

$$d\alpha(v_0^2 z \sec^2\alpha - gz^2 \sec^2\alpha \tan\alpha) = dz(gz \sec^2\alpha - v_0^2 \tan\alpha).$$

$$\frac{d\alpha}{dz} = 0$$

if

$$gz \sec^2\alpha - v_0^2 \tan\alpha = 0;$$

i. e., if

$$z = \frac{v_0^2}{g} \sin\alpha \cos\alpha.$$

Then

$$2hv_0^2 = \frac{2v_0^4}{g} \sin^2\alpha - \frac{v_0^4}{g} \sin^2\alpha = \frac{v_0^4}{g} \sin^2\alpha;$$

i. e.,

$$\sin^2\alpha = \frac{2gh}{v_0^2},$$

and therefore,

$$z = \sqrt{\frac{2h}{g}(v_0^2 - 2gh)}.$$

According to the given conditions,

$$v_0^2 = 2g \times 2h = 4gh,$$

so that $\alpha = 45°$, $z = 2$h.

The stone is at the highest point of its trajectory when it strikes, whatever the value of v_0.

We may also solve the problem as follows:

If the stone strikes the wall with a velocity v_1, at an angle β with the horizontal, the energy used up by the blow will be $\frac{1}{2}mv_1^2 \cos^2\beta$, where $v_1^2 = v_0^2 - 2gh$, and $v_1 \cos\beta = v_0 \cos\alpha$. v_0 and h being given, it is necessary for the greatest effect that $\beta = 0$. Then $v_1 = v_0 \cos\alpha$, $v_0^2 \sin^2\alpha = 2gh$, $\sin^2\alpha = 2gh/v_0^2$, and since $2hv_0^2 = 2v_0^2 z \tan\alpha - gz^2 \sec^2\alpha$,

$$2h = 2z\sqrt{\frac{2gh}{v_0^2 - 2gh}} - gz^2 \frac{v_0^2}{v_0^2 - 2gh}, \qquad \text{or} \qquad z^2 - 2\sqrt{\frac{2h}{g}(v_0^2 - 2gh)} \cdot z + \frac{2h}{g}(v_0^2 - 2gh) = 0;$$

i. e.,

$$z = \sqrt{\frac{2h}{g}(v_0^2 - 2gh)},$$

as before.

Also solved by MARCUS SKARSTEDT.